NON-EUCLIDEAN GEOMETRIES

A Journey through the Curvature of Space

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1. INTRODUCTION

For more than two millennia, humanity believed that the geometry described by Euclid was the only possible truth, the perfect description of the space in which we live. The walls of our houses, the sheets of paper, the cultivated fields, everything seemed to obey Euclidean laws.

However, in the nineteenth century, some mathematicians dared to question one of the most fundamental postulates of this geometry. This boldness opened the door to a new universe of mathematical possibilities and, later, to a new physical understanding of the cosmos.

Non-Euclidean geometries are not a mere academic curiosity. They represent one of the greatest revolutions in human thought, comparable to the discovery that the Earth is not flat or that the Sun does not revolve around us.

2. EUCLIDEAN GEOMETRY AND THE FIFTH POSTULATE

Euclid of Alexandria, who lived around 300 B.C., wrote a work called "The Elements." In this book, which for centuries was the second most read after the Bible, Euclid organized all the geometric knowledge of his time in a logical and deductive way.

Euclid's method was simple but powerful:

1. Definitions: Explain the meaning of the terms (point, line, circle, etc.)

2. Axioms or Postulates: Truths accepted without the need for demonstration

3. Theorems: Propositions demonstrated from postulates

Euclid established five postulates:

Postulate 1: A line can be drawn between any two points.

Postulate 2: One can extend a finite line continuously in a straight line.

Postulate 3: A circle can be drawn with any center and any radius.

Postulate 4: All right angles are equal.

Postulate 5: If a line, when intersecting two others, forms internal angles on the same side whose sum is less than two right angles, then these two lines, if prolonged indefinitely, meet on that side.

This fifth postulate was better known in a simplified version, equivalent to the original:

"For a point outside a line, a single line parallel to the given line passes."

It is the famous Postulate of the Parallels.

3. THE MILLENNIAL PROBLEM

Since antiquity, mathematicians have felt discomfort with this fifth postulate. Unlike the other four, which are simple and intuitive, the fifth seemed more like a theorem than a self-evident truth.

For more than 2000 years, countless mathematicians have tried to demonstrate that the fifth postulate could be deduced from the first four. All failed.

These attempts included:

· Ptolemy (second century)

· Proclus (5th century)

· Nasir al-Din al-Tusi (13th September)

· John Wallis (17th century)

· Giovanni Saccheri (sec. XVIII)

Saccheri even tried a demonstration for absurdity: he denied the postulate of parallels and tried to find a contradiction. But, to his astonishment, he found no contradiction. He developed several theorems that were strange and different from the usual geometry, but all consistent. Unfortunately, Saccheri did not believe what he had discovered and forced a wrong conclusion, saying that these results were "repugnant to the nature of the line".

The world was not yet ready for revolution.

4. THE NINETEENTH-CENTURY REVOLUTION

In the nineteenth century, three mathematicians, working independently and with no knowledge of each other's work, came up with the same revolutionary idea:

"What if the parallel postulate is simply false? What if, instead of a single parallel, there are infinite, or none?"

These three geniuses were:

Carl Friedrich Gauss (1777-1855)

German, considered one of the greatest mathematicians of all time. Gauss developed a non-Euclidean geometry but, fearful of criticism and controversy, never published his results. He only revealed them in letters and personal notes.

Nikolai Ivanovich Lobachevsky (1792-1856)

Russian mathematician who published the first work on non-Euclidean geometry in 1829. He called it "Imaginary Geometry". It was ridiculed for years.

János Bolyai (1802-1860)

Hungarian mathematician who developed his non-Euclidean geometry in 1823. His father, also a mathematician and friend of Gauss, sent the work to Gauss, who replied:

"I cannot praise this work, for to do so would be to praise myself. All of its content coincides with my own meditations."

János Bolyai was so discouraged that he never published anything in mathematics again.

These three created what we now call Hyperbolic Geometry.

Later, Bernhard Riemann (1826-1866), a student of Gauss, created a second type of non-Euclidean geometry: Elliptical Geometry.

5. HYPERBOLIC GEOMETRY

The Postulate

In hyperbolic geometry, the fifth postulate is replaced by:

"By a point outside a line, infinite lines pass that never find the given line."

A Curvature

Hyperbolic geometry is the geometry of surfaces with negative curvature. The easiest way to visualize it is to think of a horse saddle or a Pringles potato. On such a surface, the (geodesic) lines behave very differently.

Key features

1. Triangles

· The sum of the interior angles of a triangle is always less than 180°

· The larger the triangle, the further the sum moves away from 180°

· Similar triangles are always congruent (there is no scale)

2. Circles

· The circumference of a circle grows exponentially with radius

· Area of a circle: much larger than in Euclidean geometry

3. Parallel Lines

· There are two types of parallels: those that approach asymptotically and those that move away

· The concept of parallelism is much more complex

4. Angles

· Parallelism angle varies with distance

· The farther away the point, the smaller the parallelism angle

Representation Models

Because it is difficult to imagine a hyperbolic surface, mathematicians have created models to represent it:

Poincaré model (disk): Represents the hyperbolic plane within a circle. "Lines" are arcs of a circle that intersect the edge at right angles. The distances are distorted: near the edge, the distances seem small but are huge.

Klein model: Similar, but the lines are chords of the circle.

Hyperboloid Model: Represents the hyperbolic geometry on a funnel-shaped surface (hyperboloid) in three-dimensional space.

6. ELLIPTICAL (OR RIEMANN) GEOMETRY

The Postulate

In elliptic geometry, the fifth postulate is replaced by:

"For a point outside a line, no parallel line passes. All the straights intersect."

A Curvature

Elliptic geometry is the geometry of surfaces with positive curvature. The simplest example is the surface of a sphere.

Key features

1. Triangles

· The sum of the internal angles of a triangle is always greater than 180°

· On Earth, a triangle with a vertex at the North Pole and two points at the Equator can have a sum of angles up to 270°

· The larger the triangle, the larger the sum

2. Retas (Geodesic)

· The "lines" are the maximum circles (such as the Equator or the lines of longitude)

· There are no infinite straights: by always moving forward, we return to the starting point

· Two straight lines (maximum circles) always intersect at two opposite points

3. Distances

· Space is finite but without borders (like the surface of the Earth)

· The circumference of a circle is smaller than in Euclidean geometry

4. Versions of Elliptical Geometry

There are two main versions:

· Spherical Geometry: Considers that antipodean points are distinct. It is the geometry of the Earth's surface.

· Elliptical geometry (proper): Identifies antipodean points as being the same point. It is more abstract but mathematically more elegant.

KEY DIFFERENCES

Detailed Comparison Table

Euclidean Hyperbolic Elliptical Feature

Curvature Zero (Flat) Negative Positive

Parallel by a point An infinite None

Sum of triangle angles = 180° < 180° > 180°

Circumference/radius ratio $C = 2\pi r$ $C > 2\pi r$ $C < 2\pi r$

Area of triangle Proportional to base × height Proportional to angular deficit Proportional to angular excess

Pythagoras' theorem $a^2 + b^2 = c^2$ $a^2 + b^2 < c^2$ $a^2 + b^2 > c^2$

Finite Infinite Infinite Lines (closed)

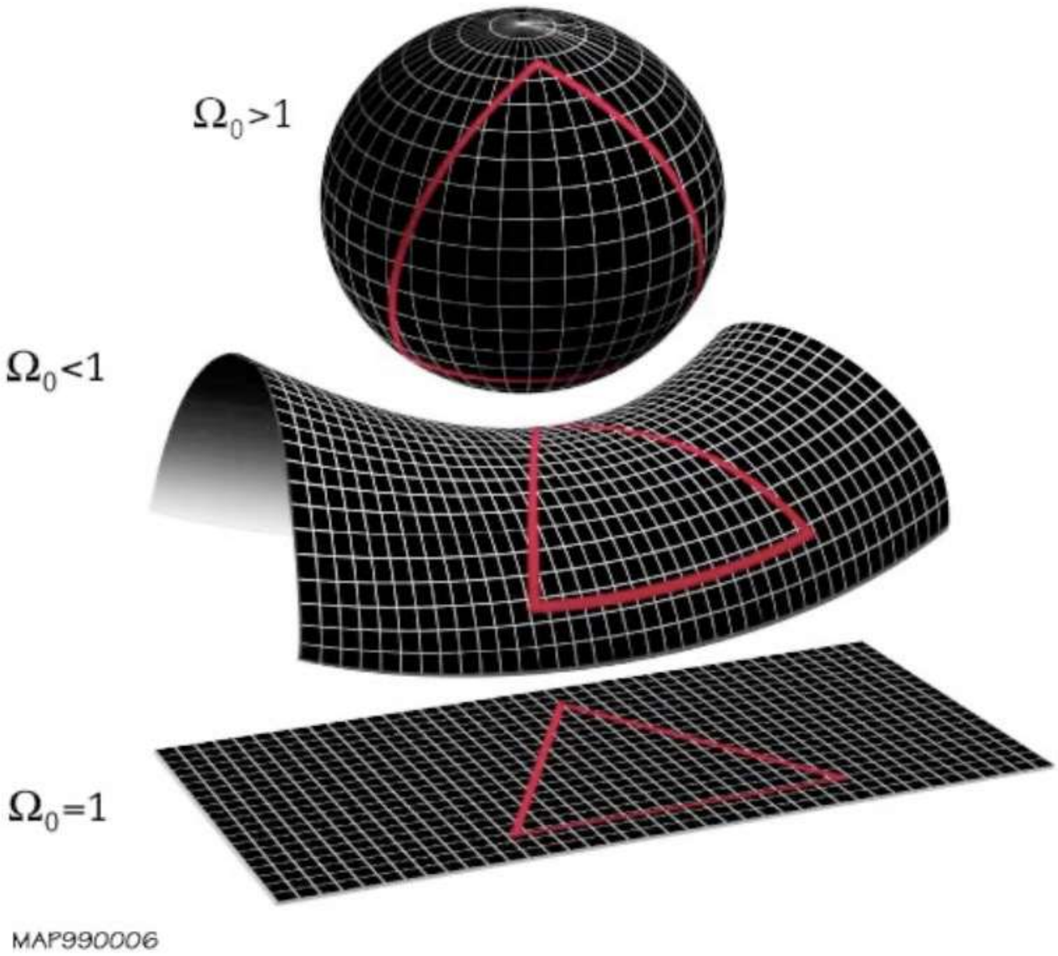
Unlimited and infinite space Unlimited and infinite Finite but without borders

Visual Examples

· Euclidean: A perfectly flat sheet of paper

· Hyperbolic: A curly lettuce, a coral, a horse saddle

· Elliptical: An orange, a balloon, the surface of the Earth



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8. EINSTEIN'S GENERAL RELATIVITY

The discovery of non-Euclidean geometries was, for decades, considered an abstract mathematical curiosity, without any application in the real world. Everything changed in 1915.

Einstein's Revolution

Albert Einstein, when developing the Theory of General Relativity, realized that he needed a mathematics that described a curved space. His mathematician friend, Marcel Grossmann, introduced him to the work of Riemann and other non-Euclidean geometers.

Einstein discovered that:

1. Gravity is bend

Gravity is not a force, as Newton thought. It is the effect of the curvature of space-time. Massive objects (such as the Sun or Earth) curve the space around them, and this curvature tells other objects how to move.

2. Spacetime is curved

We live in a four-dimensional space-time (three spatial dimensions + one temporal) that is curved. The geometry of the universe is therefore non-Euclidean.

3. Matter tells space how to bend

The presence of mass and energy determines the curvature of space-time.

4. Space tells matter how to move

The curvature of space-time determines the trajectories of objects (geodesics).

Experimental Confirmations

Einstein's theory has been confirmed several times:

· 1919: Solar eclipse confirms starlight is deflected by the curvature of space around the Sun

· GPS: GPS satellites have to correct for the effects of general (and special) relativity to give correct positions

· Gravitational waves: Detected in 2015, they are ripples in the curvature of space-time

What Geometry for the Universe?

Cosmologists are still studying what the overall geometry of the universe is:

· Flat Universe (Euclidean): Expansion Continues Forever, Critical Speed

· Hyperbolic Universe (Negative Curvature): Expansion Accelerates, Open Universe

· Elliptical Universe (Positive Curvature): Expansion Can Stop and Contract, Closed Universe

Current measurements (Planck mission, etc.) suggest that the universe is flat or very close to flat, with a small margin of error. But the issue is not completely closed.

9. PRACTICAL APPLICATIONS

GPS - Global Positioning System

GPS is perhaps the most tangible application of general relativity in our day-to-day lives:

· GPS satellites are in orbit, in a weaker gravitational field than on the Earth's surface

· Due to general relativity, time passes slightly faster on satellites

· Without relativistic corrections (which are non-Euclidean corrections), GPS would accumulate an error of several kilometers per day

Air and Sea Navigation

Airplanes and ships do not sail in a straight line (Euclidean), but rather in geodesics on the spherical surface of the Earth (elliptical geometry). The shortest route between Lisbon and New York is not a straight line on a flat map, but rather a maximum circle arc.

Art and Architecture

Graphic artist M.C. Escher created several works based on hyperbolic geometry, such as "Circle Limit IV" (1960), where figures repeat on a disk, decreasing in size as they approach the edge, perfectly illustrating Poincaré's model.

Cosmology

All modern cosmology uses non-Euclidean geometry to describe the evolution and structure of the universe.

Pure Mathematics

Non-Euclidean geometries are fundamental in areas such as:

· Topology

· Group theory

· Complex analysis

· Differential geometry

10. CONCLUSION

Non-Euclidean geometries represent one of the greatest intellectual revolutions in human history. They showed that mathematics is not an absolute truth discovered in nature, but rather a human construct based on axioms that we can choose from.

What we learned:

1. Plurality of truth: There are several geometries, all mathematically valid and consistent.

2. Space is not necessarily Euclidean: The geometry of the physical universe is determined by the distribution of mass and energy.

3. Mathematics anticipates physics: Non-Euclidean geometries were discovered nearly a century before they found application in general relativity.

4. Intellectual humility: For two millennia, humanity has been "right" but incomplete. Euclidean geometry is not wrong; it is just a particular case (zero curvature) of a wider set of possible geometries.

Euclidean Geometry: It is still valid for our daily lives. If you want to build a house, plant a field, or draw a table, use Euclid.

Hyperbolic Geometry: Useful for describing certain mathematical spaces and perhaps the universe on a large scale.

Elliptical Geometry: Essential for navigation on Earth and for describing the curved space-time of general relativity.

"The universe is not only stranger than we suppose, it is stranger than we can suppose." – J.B.S. Haldane

Non-Euclidean geometries have opened our minds to this strangeness, preparing us to understand a curved, finite but boundless cosmos, where parallel lines can meet and triangles can add up to more than 180°.

After all, we live in a non-Euclidean reality, even if our senses, limited to the flat space of everyday life, insist on showing us the opposite.